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LETTER TO THE EDITOR

The Fermion-ladder models: extensions of the Hubbard model with η-pairing

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Abstract. We propose two-leg fermion-ladder models for the $SU(2|2)$ and $SU(4)$ cases. The former is exactly the extended Hubbard model proposed by Essler, Korepin and Schoutens. The latter is a new model also with η-pairing symmetry which is important for superconductivity. This new extension of the Hubbard model can be solved exactly.

Since the discovery of high-temperature superconductivity, much more attention has been paid to the theoretical mechanism for such phenomena. Most proposals concern the Hubbard model and the t–J model [1]. Yang [2] mentioned the importance of the η-pairing mechanism and the property of off-diagonal long-range order (ODLRO) for the eigenfunctions in superconductivity. Essler *et al* [3] proposed an extended Hubbard model with η-pairing symmetry. They showed that the states of this extended Hubbard model exhibit ODLRO and is thus superconducting.

Recently, both in experiment and theory, there has also been growing interest in spin ladders for their relevance to some quasi-one-dimensional materials, which under hole doping may show superconductivity [4]. Many spin-ladder models and fermion-ladder models including t–J ladders and Hubbard ladders have been proposed [5].

In this letter, motivated by the the construction of spin-ladder models, we study the coupled fermion models. We construct the most simple fermion-ladder models for the $SU(2|2)$ and $SU(4)$ cases. The first one gives the extended Hubbard model which has already been studied [3], the second is a new extended Hubbard model which also has the symmetry of η-pairing and its eigenfunctions possess ODLRO.

Generally, we will concentrate on the integrable models which can be solved exactly. We first start from the integrable one-dimensional fermion chain with $SU(1|1)$ symmetry. Electrons on a lattice are described by canonical Fermi operators satisfying $\{a_i^\dagger, a_j\} = \delta_{ij}$. The Fock vacuum state $|0\rangle$ satisfies $a_i|0\rangle = 0$. The number operator for electrons on site i is denoted by $n_{i,a} = a_i^\dagger a_i$. The Hamiltonian of an L-site fermion chain is written as

$$H_a = \sum_{\langle jk\rangle}[a_j^\dagger a_k + a_k^\dagger a_j - n_{j,a} - n_{k,a}] \tag{1}$$

where summation is taken for nearest neighbours. Similarly, we suppose that there is another fermion chain with different fermion operators $b_i^\dagger, b_i$ and number operator $n_{i,b}$. The two kinds

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of Fermi operators still also satisfy the anti-commutation relation. We thus can construct the two-leg fermion-ladder model as

$$\begin{aligned} H_{ab} = t \sum_{\langle jk\rangle} [a_j^\dagger a_k + a_k^\dagger a_j - n_{j,a} - n_{k,a} + b_j^\dagger b_k + b_k^\dagger b_j - n_{j,b} - n_{k,b}] \\ + J \sum_{\langle jk\rangle} [a_j^\dagger a_k + a_k^\dagger a_j - n_{j,a} - n_{k,a}] \times [b_j^\dagger b_k + b_k^\dagger b_j - n_{j,b} - n_{k,b}] \\ + U \sum_{j=1}^{L} n_{j,a} n_{j,b} + \mu_a \sum_{j=1}^{L} n_{j,a} + \mu_b \sum_{j=1}^{L} n_{j,b}. \end{aligned} \tag{2}$$

This Hamiltonian does not include all coupled terms between the two fermion chains, but we will still reduce this Hamiltonian to a more special case. Let $t = -1$, $J = -1$, $\mu_a = \mu_b = -U/2$, up to a constant, this Hamiltonian is equivalent to the extended Hubbard model proposed in [3]. Here we use more familiar notation; actually we express these two kinds of Fermi operators as one kind of Fermi operator with spin-down and spin-up respectively: let $a_j^\dagger = c_{j,\uparrow}^\dagger$, $a_j = c_{j,\uparrow}$, $b_j^\dagger = c_{j,\downarrow}^\dagger$, $b_j = c_{j,\downarrow}$, and similarly we denote $n_{j,a} = n_{j,\uparrow}$, $n_{j,b} = n_{j,\downarrow}$, and $n_j = n_{j,\uparrow} + n_{j,\downarrow}$. The anticommutation relations between Fermi operators can be expressed as $\{c_{i,\sigma}^\dagger, c_{j,\tau}\} = \delta_{ij}\delta_{\sigma\tau}$. We can write the fermion-ladder model as

$$H_{eks} = -H_{eks}^0 + U \sum_{j=1}^{L} (n_{j,\uparrow} - \tfrac{1}{2})(n_{j,\downarrow} - \tfrac{1}{2}) \tag{3}$$

where the two coupled fermion chains are presented as

$$H_{eks}^0 = \sum_{\langle jk\rangle} [c_{j,\uparrow}^\dagger c_{k,\uparrow} + c_{k,\uparrow}^\dagger c_{j,\uparrow} - n_{j,\uparrow} - n_{k,\uparrow} + 1] \times [c_{j,\downarrow}^\dagger c_{k,\downarrow} + c_{k,\downarrow}^\dagger c_{j,\downarrow} - n_{j,\downarrow} - n_{k,\downarrow} + 1]. \tag{4}$$

We present the above Hamiltonian in detail as

$$\begin{aligned} H_{eks}^0 = \sum_{\langle jk\rangle} c_{k,\uparrow}^\dagger c_{j,\uparrow}(1 - n_{j,\downarrow} - n_{k,\downarrow}) + c_{j,\uparrow}^\dagger c_{k,\uparrow}(1 - n_{j,\downarrow} - n_{k,\downarrow}) + c_{k,\downarrow}^\dagger c_{j,\downarrow}(1 - n_{j,\uparrow} - n_{k,\uparrow}) \\ + c_{j,\downarrow}^\dagger c_{k,\downarrow}(1 - n_{j,\uparrow} - n_{k,\uparrow}) + \tfrac{1}{2}(n_j - 1)(n_k - 1) + c_{j,\uparrow}^\dagger c_{j,\downarrow}^\dagger c_{k,\downarrow} c_{k,\uparrow} \\ + c_{j,\downarrow} c_{j,\uparrow} c_{k,\uparrow}^\dagger c_{k,\downarrow}^\dagger - \tfrac{1}{2}(n_{j,\uparrow} - n_{j,\downarrow})(n_{k,\uparrow} - n_{k,\downarrow}) - c_{j,\downarrow}^\dagger c_{j,\uparrow} c_{k,\uparrow}^\dagger c_{k,\downarrow} \\ - c_{j,\uparrow}^\dagger c_{j,\downarrow} c_{k,\downarrow}^\dagger c_{k,\uparrow} + (n_{j,\uparrow} - \tfrac{1}{2})(n_{j,\downarrow} - \tfrac{1}{2}) + (n_{k,\uparrow} - \tfrac{1}{2})(n_{k,\downarrow} - \tfrac{1}{2}). \end{aligned} \tag{5}$$

This Hamiltonian is simply the model proposed by Essler *et al* and has the symmetry of $SU(2|2)$. It was studied in detail in [6].

Now, let us examine the $SU(4)$ case. Generally, the $SU(2)$ fermion chain is presented as $H_a' = \sum_{\langle jk\rangle} [a_j^\dagger a_k + a_k^\dagger a_j + 2n_{j,a}n_{k,a} - n_{j,a} - n_{k,a}]$. Similarly to the above method, we can construct a two-leg fermion ladder with the $SU(4)$ symmetry

$$H = -H^0 + U \sum_{j=1}^{L} (n_{j,\uparrow} - \tfrac{1}{2})(n_{j,\downarrow} - \tfrac{1}{2}) \tag{6}$$

where $H^0 = \sum_{\langle jk\rangle} H_{jk}^0$ and we define

$$\begin{aligned} H_{jk}^0 = [c_{j,\uparrow}^\dagger c_{k,\uparrow} + c_{k,\uparrow}^\dagger c_{j,\uparrow} + 2n_{j,\uparrow}n_{k,\uparrow} - n_{j,\uparrow} - n_{k,\uparrow} + 1] \\ \times [c_{j,\downarrow}^\dagger c_{k,\downarrow} + c_{k,\downarrow}^\dagger c_{j,\downarrow} + 2n_{j,\downarrow}n_{k,\downarrow} - n_{j,\downarrow} - n_{k,\downarrow} + 1]. \end{aligned} \tag{7}$$

We write explicitly this new Hamiltonian in the form of $SU(4)$ generators

$$H_{jk}^0 = \sum_{\sigma=\uparrow,\downarrow} [(c_{j,\sigma}^\dagger c_{k,\sigma} + c_{k,\sigma}^\dagger c_{j,\sigma})(1-n_{j,-\sigma})(1-n_{k,-\sigma}) + (c_{j,\sigma}^\dagger c_{k,\sigma} + c_{k,\sigma}^\dagger c_{j,\sigma}) n_{j,-\sigma} n_{k,-\sigma}]$$
$$+ c_{j,\uparrow}^\dagger c_{j,\downarrow}^\dagger c_{k,\downarrow} c_{k,\uparrow} + c_{j,\downarrow} c_{j,\uparrow} c_{k,\uparrow}^\dagger c_{k,\downarrow}^\dagger - c_{j,\downarrow}^\dagger c_{j,\uparrow} c_{k,\uparrow}^\dagger c_{k,\downarrow} - c_{j,\uparrow}^\dagger c_{j,\downarrow} c_{k,\downarrow}^\dagger c_{k,\uparrow}$$
$$+ \tfrac{1}{2}(n_{j,\uparrow} - n_{j,\downarrow})(n_{k,\uparrow} - n_{k,\downarrow}) + \tfrac{1}{2}(n_j - 1)(n_k - 1)$$
$$+ 4(n_{j,\uparrow} - \tfrac{1}{2})(n_{j,\downarrow} - \tfrac{1}{2})(n_{k,\uparrow} - \tfrac{1}{2})(n_{k,\downarrow} - \tfrac{1}{2}) + \tfrac{1}{4}. \tag{8}$$

The Hamiltonian H^0, as in the $SU(2|2)$ case, is invariant under spin reflection $c_{j,\uparrow} \leftrightarrow c_{j,\downarrow}$. However, unlike the $SU(2|2)$ case, it does not have the property of particle–hole replacement $c_{j,\sigma}^\dagger \leftrightarrow c_{j,\sigma}$ invariance. There are four kinds of state at a given site: $|0\rangle_i$, $|\uparrow\rangle_i = c_{i,\uparrow}^\dagger |0\rangle_i$, $|\downarrow\rangle_i = c_{i,\downarrow}^\dagger |0\rangle_i$, $|\uparrow\downarrow\rangle_i = c_{i,\downarrow}^\dagger c_{i,\uparrow}^\dagger |0\rangle$, two of them are fermionic and the other two are bosonic. The last state represents an electron pair localized on a single lattice site; the pair are called localons, and this is considered to be the mechanism to form 'Cooper pairs' [3].

The new Hamiltonian commutes with η-pairings $\eta^\dagger = \sum_j^L c_{j,\downarrow}^\dagger c_{j,\uparrow}^\dagger$ and $\eta = \sum_j^L c_{j,\uparrow} c_{j,\downarrow}$. It is argued that η-pairs is a rather typical phenomenon in superconductivity [6]. In terms of electronic operators $c_{\boldsymbol{k}\sigma}^\dagger$ in momentum space, $c_{j,\sigma}^\dagger = \frac{1}{\sqrt{L}} \sum_{\boldsymbol{k}} \mathrm{e}^{\mathrm{i}\boldsymbol{k}j} c_{\boldsymbol{k}\sigma}^\dagger$, we find $\eta^\dagger = \sum_{\boldsymbol{k}} c_{\boldsymbol{k}\downarrow}^\dagger c_{-\boldsymbol{k}\uparrow}^\dagger$ which is simply the BCS order parameter, and similarly for the case $\eta = \sum_{\boldsymbol{k}} c_{\boldsymbol{k}\uparrow} c_{-\boldsymbol{k}\downarrow}$.

The η-pairs form a $SU(2)$ algebra. The generators $\eta_j^\dagger = c_{j,\downarrow}^\dagger c_{j,\uparrow}^\dagger$, $\eta_j = c_{j,\uparrow} c_{j,\downarrow}$ and $\eta_j^z = -\frac{1}{2} n_j + \frac{1}{2}$ satisfy the relations $[\eta_j, \eta_j^\dagger] = 2\eta_j^z$, $[\eta_j^\dagger, \eta_j^z] = \eta_j^\dagger$, $[\eta_j, \eta_j^z] = -\eta_j$. Similarly, the spin operators $S_j = c_{j,\uparrow}^\dagger c_{j,\downarrow}$, $S_j^\dagger = c_{j,\downarrow}^\dagger c_{j,\uparrow}$ and $S_j^z = \frac{1}{2}(n_{j,\uparrow} - n_{j,\downarrow})$ also satisfy $SU(2)$ algebra $[S_j, S_j^\dagger] = 2S_j^z$, $[S_j^\dagger, S_j^z] = S_j^\dagger$, $[S_j, S_j^z] = -S_j$. These generators are Grassmann-even (bosonic), and generator $X_j = (n_{j,\uparrow} - \frac{1}{2})(n_{j,\downarrow} - \frac{1}{2})$ is also a Grassmann-even operator. Here we introduce eight Grassmann-odd (fermionic) generators $Q_{j,\sigma} = (1 - n_{j,-\sigma}) c_{j,\sigma}$, $Q_{j,\sigma}^\dagger = (1 - n_{j,-\sigma}) c_{j,\sigma}^\dagger$, $\tilde{Q}_{j,\sigma} = n_{j,-\sigma} c_{j,\sigma}$, $\tilde{Q}_{j,\sigma}^\dagger = n_{j,-\sigma} c_{j,\sigma}^\dagger$, with $\sigma = \uparrow, \downarrow$ representing spin-up and spin-down, respectively. In terms of these generators, the Hamiltonian H_{jk}^0 is written as

$$H_{jk}^0 = \sum_{\sigma=\uparrow,\downarrow} [Q_{j,\sigma}^\dagger Q_{k,\sigma} + Q_{k,\sigma}^\dagger Q_{j,\sigma} + \tilde{Q}_{j,\sigma}^\dagger \tilde{Q}_{k,\sigma} + \tilde{Q}_{k,\sigma}^\dagger \tilde{Q}_{j,\sigma}]$$
$$+ \eta_j^\dagger \eta_k + \eta_j \eta_k^\dagger + 2\eta_j^z \eta_k^z - S_j^\dagger S_k - S_j S_k^\dagger + 2S_j^z S_k^z + 4X_j X_k + \tfrac{1}{4}. \tag{9}$$

Because we use the graded method, the Hamiltonian loses the $SU(4)$ invariant property: for example, $[S^\dagger, H^0] \neq 0$, with $S^\dagger = \sum_j^L S_j^\dagger$. The one-dimensional Hamiltonian is equal to a graded $SU(4)$ permutation operator and can be diagonalized by using the graded Bethe ansatz method. The energy of the Hamiltonian H^0 is given by [8]

$$E = L - \sum_{j=1}^{N} \frac{1}{\lambda_j^2 + \frac{1}{4}} \tag{10}$$

where λ_j satisfy the following Bethe ansatz equations:

$$\left(\frac{\lambda_k - \frac{\mathrm{i}}{2}}{\lambda_k + \frac{\mathrm{i}}{2}} \right)^L = \prod_{j=1, \neq k}^{N} \frac{\lambda_k - \lambda_j - \mathrm{i}}{\lambda_k - \lambda_j + \mathrm{i}} \prod_{l=1}^{N^{(1)}} \frac{\lambda_l^{(1)} - \lambda_k - \frac{\mathrm{i}}{2}}{\lambda_l^{(1)} - \lambda_k + \frac{\mathrm{i}}{2}} \tag{11}$$

$$\prod_{j=1}^{N} \frac{\lambda_k^{(1)} - \lambda_j + \frac{\mathrm{i}}{2}}{\lambda_k^{(1)} - \lambda_j - \frac{\mathrm{i}}{2}} = \prod_{l=1}^{N^{(1)}} \frac{\lambda_l^{(1)} - \lambda_k^{(1)} - \mathrm{i}}{\lambda_k^{(1)} - \lambda_l^{(1)} - \mathrm{i}} \prod_{j=1}^{N^{(2)}} \frac{\lambda_k^{(1)} - \lambda_j^{(2)} - \frac{\mathrm{i}}{2}}{\lambda_j^{(2)} - \lambda_k^{(1)} - \frac{\mathrm{i}}{2}} \tag{12}$$

$$\prod_{l=1,\neq k}^{N^{(2)}} \frac{\lambda_l^{(2)} - \lambda_k^{(2)} + \mathrm{i}}{\lambda_l^{(2)} - \lambda_k^{(2)} - \mathrm{i}} = \prod_{j=1}^{N^{(1)}} \frac{\lambda_k^{(2)} - \lambda_j^{(1)} - \frac{\mathrm{i}}{2}}{\lambda_j^{(1)} - \lambda_k^{(2)} - \frac{\mathrm{i}}{2}} \tag{13}$$

and k takes values, $1, \ldots, N$, $1 \ldots N^{(1)}$ and $1, \ldots, N^{(2)}$, respectively, in the above Bethe ansatz equations.

The Hamiltonian commutes with the generator $X = \sum_{j=1}^{L} X_j$, so the interaction term in (6) with coupling constant U does not break the integrability of the Hamiltonian. Generally, we can also add such terms as chemical potential and magnetic field in the Hamiltonian which does not change the integrability of the model.

We define the η-pairing generators $\eta = \sum_{j=1}^{L} \eta_j$, $\eta^\dagger = \sum_{j=1}^{L} \eta_j^\dagger$ and $\eta^z = \sum_{j=1} \eta_j^z$ which constitute the $SU(2)$ algebra [3]. We find that the Hamiltonian commutes with those generators $[H, \eta] = [H, \eta^\dagger] = 0$. The state $(\eta^\dagger)^N|0\rangle$ is an eigenstate of the Hamiltonian (6) with eigenvalue $E = \frac{UL}{4} - M$, here we assume that the total number of nearest-neighbour links $\langle jk \rangle$ in the lattice is M. As shown in [3], this eigenstate posseses ODLRO. Explicitly, the off-diagonal matrix element of the reduced density matrix reads

$$\frac{\langle 0|\eta^N c_{k,\downarrow}^\dagger c_{k,\uparrow}^\dagger c_{l,\uparrow} c_{l,\downarrow} (\eta^\dagger)^N|0\rangle}{\langle 0|\eta^N (\eta^\dagger)^N|0\rangle} = \frac{N(L-N)}{L(L-1)}. \tag{14}$$

Here we assume $k \neq l$, and find the relation $\langle 0|\eta^N(\eta^\dagger)^N|0\rangle = N!L \times \cdots \times (L-N+1)$ useful in the above calculation. We find that the off-diagonal matrix element is constant for large distance $|k - l|$ which means that the eigenstate has the ODLRO property. Here we have shown, as in the $SU(2|2)$ case, that the state $(\eta)^N|0\rangle$ which possesses ODLRO is also an eigenstate of the new Hamiltonian (6).

We can propose a more general extension of the Hubbard model, besides the original terms [6, 7]: we add an extra term $W(c_{j,\sigma}^\dagger c_{k,\sigma} + c_{k,\sigma}^\dagger c_{j,\sigma}) n_{j,-\sigma} n_{k,-\sigma}$ in the Hamiltonian.

The n-leg integrable fermion chains are assumed as follows:

$$H^0 = \sum_{\langle jk \rangle} \prod_{\sigma=1}^{n} [c_{j,\sigma}^\dagger c_{k,\sigma} + c_{k,\sigma}^\dagger c_{j,\sigma} - n_{j,\sigma} - n_{k,\sigma} + 1] \tag{15}$$

$$H^0 = \sum_{\langle jk \rangle} \prod_{\sigma=1}^{n} [c_{j,\sigma}^\dagger c_{k,\sigma} + c_{k,\sigma}^\dagger c_{j,\sigma} + 2n_{j,\sigma} n_{k,\sigma} - n_{j,\sigma} - n_{k,\sigma} + 1]. \tag{16}$$

The first one has $SU(2^{n-1}|2^{n-1})$ symmetry, and the second is the $SU(2^n)$ case. The Hamiltonian can also be diagonalized by the Bethe ansatz method.

In conclusion, we have proposed fermion-ladder models for two cases. For two-leg fermion-ladder models, we have obtained extensions of the Hubbard model with η-pairing. The Hamiltonians were the $SU(2|2)$ and $SU(4)$ cases. The $SU(4)$ case was a new model, and its eigenstate was found to possess the ODLRO property. The n-leg fermion ladders were also conjectured.

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